

ON SUMS, PRODUCTS , AND THE MULTIDIMENSIONAL FALCONER PROBLEM

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§0 INTRODUCTION

The Falconer distance conjecture says that if a set $E \subset \mathbb{R}^d$ has Hausdorff dimension at least $\frac{d}{2}$ then its distance set $D = \{|x - y|, x, y \in E\}$ has Hausdorff dimension at least 1. This problem has received a great deal of attention in recent years. Wolff [W] got a partial result for $d = 2$ showing the conjecture holds if the dimension of the set is at least $\frac{4}{3}$. Erdogan [E] extended Wolff's method to obtain partial results in higher dimensions.

In this paper, we prove essentially

Theorem 0.1 There exists $\epsilon_d > 0$ depending only on the dimension d so that if E has Hausdorff dimension at least $\frac{d}{2}$ then D has Hausdorff dimension at least $\frac{1}{2} + \epsilon_d$.

To be precise, we prove Theorem 1.1, a multilinear version of Theorem 0.1. However, the interested reader will see that Theorem 1.1 implies Theorem 0.1 following the treatment in [KT] (section 9). In the case $d = 2$, this theorem is the combination of the results of [KT] and [B], so that technically the present paper is merely a generalization of sections 5 and 6 of [KT] to higher dimensions. This involves a few technical issues, but essentially uses the same ideas present there.

Largely, the inspiration for this paper is a preprint of Iosevich and Rudnev [IR], in which the authors assert that a set having Hausdorff dimension $\frac{d}{2}$ but distance set of dimension $\frac{1}{2}$ must have a lot of additive structure. This is obtained from a Fourier analytic estimate. We wrote this paper because we felt that the authors understated the case. Not only does the set have additive properties, but it also has multiplicative properties. Thus, *a fortiori*, apply Bourgain's sum-product estimate, the set does not exist. It is the presence of this multiplicative structure in the higher dimensional case which the present work establishes

1991 *Mathematics Subject Classification.* primary 05B99 secondary 28A75, 28A78.

The author was supported by NSF grant DMS 0432237

§1 ESTABLISHING LINEAR STRUCTURE

In this section, we will begin the proof of the “multilinear multidimensional Falconer conjecture.”

Following [KT], we define the notion of a $(\delta, \rho)_d$ set. We say that a set $A \subset \mathbb{R}^d$ is a $(\delta, \rho)_d$ set provided that if B is a ball of radius r , we have that

$$|B \cap A| \leq \frac{\delta^\rho}{r}.$$

Theorem 1.1 (multilinear, multidimensional Falconer conjecture) There is a constant $\epsilon_d > 0$ so that the following holds. Let B_1, \dots, B_{d+1} be unit balls in \mathbb{R}^d . Let E_1, \dots, E_{d+1} be $(\delta, \frac{d}{2})_d$ sets with $E_j \subset B_j$. Suppose that for every x_1, \dots, x_{d+1} with $x_j \in B_j$ we have $|(x_2 - x_1) \wedge (x_3 - x_1) \cdots \wedge (x_{d+1} - x_1)| \sim 1$. Let D be a $(\delta, \frac{1}{2})_1$ set. Let

$$X = \{(x_1, x_2, \dots, x_{d+1}) : x_j \in E_j, 1 \leq j \leq d+1, |x_k - x_j| \in D, 1 \leq k < j \leq d+1\}.$$

Then

$$|X| \leq \delta^{\frac{d(d+1)}{2} + \epsilon_d}.$$

We now say a few words about notation and the structure of the argument. Our goal will be to show that a counterexample to the multilinear, multidimensional Falconer conjecture with a small value of ϵ_d makes it possible to produce a counterexample to the Bourgain sum-product theorem with $\epsilon = C_d \epsilon_d$ for some dimensionally dependent constant C_d . We will make no effort to estimate this constant and will set up our notation accordingly. We shall denote by $A \lesssim B$, the expression $A \leq C\delta^{-\epsilon}B$, where the constant and the ϵ vary from line to line. By subtle contrast, we will use $A \lll B$ to mean that $A \leq C\delta^\epsilon B$ where the C and ϵ vary from line but the ϵ is bigger than a large multiple of any ϵ that has appeared previously. A typical argument we will make will concern a set S and a real valued function q defined on S . If we know that

$$|S| \gtrsim 1,$$

and

$$(1.1) \quad |\{x \in S : q(x) \lll 1\}| \lll 1,$$

then

$$(1.2) \quad |\{x \in S : q(x) \gtrsim 1\}| \gtrsim 1.$$

This is because the ϵ used in (1.1), while it must be large compared to all previous ϵ 's, need not be large compared to the ϵ in (1.2). We use this idea to carry out various “two ends” type arguments which show that of a certain class of configurations, most are nondegenerate. These same kinds of arguments are carried out repeatedly in [KT] by numbering constants and ϵ 's. We find the present notation simpler for both the author and the reader.

We begin with a geometrical lemma which help us to guarantee non-degeneracy.

Lemma 1.2 (Nondegeneracy) With the notation of Theorem 1.1, let $F \subset E_1$ and let $x_2 \in E_2, \dots, x_{d+1} \in E_{d+1}$. Suppose that for every $y \in F$, we have that for each $2 \leq j \leq d+1$, we have $|y - x_j| \in D$. Then

(i) If H is a $d - k$ dimensional affine subspace of \mathbb{R}^d . Then

$$|\{y \in F : \exists z \in H, |z - y| \lesssim \delta\}| \lesssim \delta^{\frac{k+d}{2}},$$

and

$$|\{y \in F : \exists z \in H, |z - y| \lll 1\}| \lll \delta^{\frac{d}{2}}.$$

(ii) If S is a sphere whose center is a point $x \in E_2$ and H is a $d - k + 1$ affine subspace of \mathbb{R}^d then

$$|\{y \in F : \exists z \in H \cap S, |z - y| \lesssim \delta\}| \lesssim \delta^{\frac{k+d}{2}},$$

and

$$|\{y \in F : \exists z \in H, |z - y| \lll 1\}| \lll \delta^{\frac{d}{2}}.$$

Proof To prove (i), we simply use

$$c(z) = (|z - x_2|, |z - x_3|, \dots, |z - x_{d+1}|)$$

as a coordinate system on Q_1 . When restricted to F these coordinates take values only in the set D . By the implicit function theorem, everywhere locally on H we may find a subset of cardinality $d - k$ of the coordinates which serve as a coordinate system.

To prove (ii), we do the same thing with

$$c(z) = (|z - x|, |z - x_3|, \dots, |z - x_{d+1}|).$$

Here the first coordinate is restricted to a single value and the others vary in D . \square

Remark 1.3 The same proof shows that $H \cap F$ is a $(\frac{d-k}{2}, \delta)_{d-k}$ set when viewed as a subset of H . The same lemma is true for any permutation of $1, \dots, k+1$ by the symmetry

of the hypotheses. A set F satisfying the hypotheses of Lemma 1.2 will be referred to as **nondegenerate**.

The following pigeonholing lemma will be used so frequently that is worth stating explicitly here:

Lemma 1.4 (Pigeonholing) Let $A \subset \mathbb{R}^{n_1}$ and $B \subset \mathbb{R}^{n_2}$ and let \sim be a relation between them. let $C = \{(a, b) : a \in A, b \in B, a \sim b\}$. Then

i) **(Chebychev)** There exists $b \in B$ so that

$$|\{a \in A : a \sim b\}| \geq \frac{|C|}{|B|}.$$

ii) **(Cauchy-Schwarz)**

$$|\{(a, b, b') : a \sim b, a \sim b'\}| \geq \frac{|C|^2}{|A|}.$$

iii) **(Hölder)**

$$|\{(a, b_1, \dots, b_d) : a \sim b_1, b_2, \dots, b_d\}| \geq \frac{|C|^d}{|A|^{d-1}}.$$

Of course Lemma 1.4 is proved by applying respectively the Chebychev, Cauchy-Schwarz, and Hölder inequalities.

Now we begin the proof of Theorem 1.1. We proceed by contradiction and assume we are given E_1, \dots, E_{d+1} which satisfy the hypotheses of Theorem 1.1 but with

$$X = \{(x_1, x_2, \dots, x_{d+1}) : x_j \in E_j, 1 \leq j \leq d+1, |x_k - x_j| \in D, 1 \leq k < j \leq d+1\},$$

we have

$$|X| \gtrapprox \delta^{\frac{d(d+1)}{2}}.$$

Lemma 1.5(Refinement to nondegenerate sets) We may find $E'_1 \subset E_1$ and $E'_2 \subset E_2$ nondegenerate so that

$$Y = \{(x_1, x_2, \dots, x_{d+1}) : x_1 \in E'_1, x_2 \in E_2^{prime}, x_j \in E_j, 3 \leq j \leq d+1, |x_k - x_j| \in D, 1 \leq k < j \leq d+1\}.$$

Then

$$|Y| \gtrapprox \delta^{\frac{d(d+1)}{2}}.$$

Proof It suffices to find E'_1 nondegenerate so that with

$$Z = \{(x_1, x_2, \dots, x_{d+1}) : x_1 \in E'_1, x_2 \in E_2, x_j \in E_j, 3 \leq j \leq d+1, |x_k - x_j| \in D, 1 \leq k < j \leq d+1\},$$

we have

$$|Z| \gtrsim \delta^{\frac{d(d+1)}{2}}.$$

Once we have done this, by symmetry, we can switch the subscripts 1 and 2 and apply the argument again.

We let

$$W = \{(x_1, x_2, \dots, x_{d+1}, y_2, \dots, y_{d+1}) : x_1 \in E_1, x_j, y_j \in E_j, 2 \leq j \leq d+1, |x_1 - x_j|, |x_1 - y_j|, |x_k - x_j|, |y_k - y_j| \in D, 2 \leq k < j \leq d+1\},$$

Applying Lemma 1.4 (ii) we see that

$$|W| \gtrsim \delta^{\frac{(2d+1)d}{2}}.$$

Now we apply Lemma 1.4 (i) to fix y_2, \dots, y_{d+1} . Letting E'_1 be the set of $x \in E_1$ so that $|x - y_j| \in D$ for all $2 \leq j \leq d+1$, we obtain the desired conclusion. \square

Our next few lemmas deal with the fact that the presence of a small distance set implies that one can find many lines highly incident to our sets. The main geometric fact to keep in mind is that the set of points in \mathbb{R}^d which are equidistant to a fixed set of d points in general position y_1, \dots, y_d is the line perpendicular to the hyperplane containing the y 's which intersects that hyperplane at the center of mass.

In the next series of lemmas for t_1, t_2 real numbers, we write $t_1 \sim t_2$ if

$$|t_1 - t_2| \lesssim \delta.$$

Lemma 1.6(Existence of lines) Given y_1, \dots, y_d in \mathbb{R}^d , we will say they are $\in EGP$ (essentially in general position) provided that the $d-1$ vector $(y_2 - y_1) \wedge (y_3 - y_1) \cdots \wedge (y_d - y_1)$ is large, specifically

$$|(y_2 - y_1) \wedge (y_3 - y_1) \cdots \wedge (y_d - y_1)| \gtrsim 1.$$

Let

$$W = \{(y_1, y_2, \dots, y_d, x) : y_1, y_2, \dots, y_d \in E'_1, EGP, x \in E'_2, |y_1 - x| \sim |y_2 - x| \sim \cdots \sim |y_d - x|\}.$$

Then

$$|W| \gtrsim \delta^{\frac{d^2+d-1}{2}}.$$

Proof In light of Lemma 1.4 (i), we can fix x_3, \dots, x_{d+1} in the conclusion of Lemma 1.5, to see that

$$(1.3) \quad |\{(x_1, x_2) \in E'_1 \times E'_2 : |x_1 - x_2| \in D\}| \gtrsim \delta^{\frac{2d}{2}}.$$

We define

$$E''_2 = \{x \in E_2 : |\{x_1 \in E'_1 : |x_1 - x| \in D\}| \gtrsim \delta^{\frac{d}{2}}\}.$$

By (1.3), we have

$$|E''_2| \gtrsim |E'_2| \gtrsim \delta^{\frac{d}{2}}.$$

For a given $y \in E''_2$ and a fixed $r \in D$, we have that

$$|\{x \in E'_1 : |x - y| \sim r\}| \lesssim \delta^{\frac{d+1}{2}},$$

by the nondegeneracy of E'_1 (Lemma 1.2 (ii)). Now since D is a $(\delta, \frac{1}{2})_1$ set, we must be able for each $y \in E''_2$ to be able to find a set $D_1 \subset D$ which is $\ggg \delta$ -separated and which has cardinality $\gtrsim \delta^{-\frac{1}{2}}$ so that for each $d \in D_1$ we have

$$(1.4) \quad |\{x \in E'_1 : |x - y| \sim d\}| \lesssim \delta^{\frac{d+1}{2}}.$$

(We made D_1 to be $\ggg \delta$ separated so that the sets in (1.4) may be taken disjoint. That is the \sim in (1.4) still uses the ϵ 's from before the \ggg , although the lower bound on the cardinality of D_1 does not.). Now we conclude that with

$$W_1 = \{(y_1, y_2, \dots, y_d, x) : y_1, y_2, \dots, y_d \in E'_1, x \in E''_2, |y_1 - x| \sim |y_2 - x| \sim \dots \sim |y_d - x|\},$$

we have

$$|W_1| \gtrsim \delta^{\frac{d^2+2d-1}{2}}.$$

This is not quite the proof of the lemma because (y_1, \dots, y_d) are not necessarily in essentially general position. However we observe that by the nondegeneracy (Lemma 1.2 (ii) again) of E' , we have for $x \in E''_2$ and $r \in D$ fixed that

$$(1.5) \quad |\{(y_1, \dots, y_d) \in (E'_1) : |x - y_j| \sim r, |(y_2 - y_1) \wedge \dots (y_d - 1)| \lll 1\}| \lesssim \delta^{\frac{d(d+1)}{2}}.$$

Thus is we define

$$W_2 = \{(y_1, y_2, \dots, y_d, x) : y_1, y_2, \dots, y_d \in E'_1, \text{notEGP}, x \in E''_2, |y_1 - x| \sim |y_2 - x| \sim \dots \sim |y_d - x|\},$$

then we have

$$|W_2| \lll \delta^{\frac{d^2+2d-1}{2}}.$$

Then we set

$$W = W_1 \setminus W_2,$$

so that we have

$$|W| \gtrsim \delta^{\frac{d^2+2d-1}{2}}.$$

□

We define U to be the set of d -tuples $(y_1, \dots, y_d) \in (E'_1)^d$ in EGP . Then

$$|U| \gtrsim \delta^{\frac{d^2}{2}}.$$

We say that $(y_1, \dots, y_d) \longrightarrow x$ (read (y_1, \dots, y_d) point at x) if

$$|y_1 - x| \sim |y_2 - x| \cdots \sim |y_d - x| \in D.$$

Lemma 1.7 (nondegeneracy of lines)

(i) Fix $(y_1, \dots, y_d) \in U$. Then

$$|\{x : (y_1, \dots, y_d) \longrightarrow x\}| \lesssim \delta^{\frac{2d-1}{2}}.$$

(ii) Fix $x \in E''_2$. Then

$$|\{(y_1, \dots, y_d) : (y_1, \dots, y_d) \longrightarrow x\}| \lesssim \delta^{\frac{d^2+d-1}{2}}.$$

Proof To prove (i), we see that $\{x : (y_1, \dots, y_d) \longrightarrow x\}$ is contained in a $(\lesssim \delta)$ -tube. Thus we obtain the inequality immediately from nondegeneracy of E'_2 .

To prove (ii), we observe that there is a collection of $\delta^{-\frac{1}{2}}$ spheres S_j centered at x so that if $(y_1, \dots, y_d) \longrightarrow x$, then y_1, \dots, y_d are all at distance $\lesssim \delta$ from S_j . Applying nondegeneracy of E'_1 and summing over j , we obtain the desired result.

□

We are in now a position to prove the analogue of the statement from [KT] which said that the discretized Furstenberg conjecture implied the bilinear Falconer conjecture.

If $x_1, x_2 \in E''_2$ we say that $x_1 \triangle x_2$ (read x_1 has more than a passing acquaintance with x_2) if there is $(y_1, \dots, y_d) \in U$ so that

$$(y_1, \dots, y_d) \longrightarrow x_1, x_2,$$

and

$$|\{x \in E''_2 : (y_1, \dots, y_d) \longrightarrow x\}| \gtrsim \delta^{\frac{2d-1}{2}},$$

and

$$|x_1 - x_2| \gtrapprox 1.$$

Lemma 1.8 (Furstenburg implies Falconer) We have the estimate

$$|\{(x_1, x_2) \in (E_2'')^2 : x_1 \triangle x_2\}| \gtrapprox \delta^d.$$

Proof We let $U' \subset U$ be the set of $u \in U$ so that

$$|\{x \in E_2'' : u \longrightarrow x\}| \gtrapprox \delta^{\frac{d^2+d-1}{2}}.$$

In light of Lemma 1.6 and Lemma 1.7 (i), we see that

$$|U'| \gtrapprox |U|,$$

and that

$$(1.5) \quad |\{(x, u) : u \in U', u \longrightarrow x\}| \gtrapprox \delta^{\frac{d^2+2d-1}{2}}.$$

Now we let $E_2^{(3)}$ be the set of x in E_2'' so that

$$|\{u \in U' : u \longrightarrow x\}| \gtrapprox \delta^{\frac{d^2+d-1}{2}}.$$

In light of (1.5) and Lemma 1.7 (ii), we have that

$$|E_2^{(3)}| \gtrapprox |E_2''|.$$

We now demonstrate that points in $E_2^{(3)}$ have more than a passing acquaintance with many points of E_2'' .

Let $x \in E_2^{(3)}$. First we observe that for any u with $u \longrightarrow x$, we have that

$$(1.6) \quad |\{x' \in E_2'' : u \longrightarrow x', |x - x'| \lll 1\}| \lll \delta^{\frac{2d-1}{2}}.$$

If (1.6) failed and $u = (y_1, \dots, y_d)$ then $\{|y_1 - x'|\} \subset D$ would fail to be a $(\delta, \frac{1}{2})_1$ set where x' runs over the same set as in (1.6). Therefore for each $x \in E_2^{(3)}$ and each $u \in U'$ with $u \longrightarrow x$, we have

$$(1.7) \quad |\{x' \in E_2'' : |x - x'| \gtrapprox 1, u \longrightarrow x'\}| \gtrapprox \delta^{\frac{2d-1}{2}}.$$

Furthermore, for each $x \in E_2^{(3)}$, by Lemma 1.4 (i) there is a distance r so that

$$|\{u = (y_1, \dots, y_d) \in U' : |x - y_1| \sim \dots \sim |x - y_d| \sim r, u \longrightarrow x\}| \gtrsim \delta^{\frac{d^2+d}{2}}.$$

Again applying Lemma 1.4(i), we can fix (y_2, \dots, y_d) so that

$$|\{y_1 : u = (y_1, \dots, y_d) \in U', u \longrightarrow x\}| \gtrsim \delta^{\frac{d+1}{2}}.$$

In this way, we obtain $\gtrsim \delta^{\frac{1-d}{2}}$, choices of y_1 which are $\ggg \delta$ separated (here for grammatical reasons we have switched the order of \ggg and \gtrsim) so that $u = (y_1, \dots, y_d) \in U'$ and $u \longrightarrow x$. In light of (1.7), we see that

$$|\{x' \in E_2'' : x \triangle x'\}| \gtrsim \delta^{\frac{d}{2}}.$$

Now integrating over $x \in E_2^{(3)}$, we obtain the desired result. \square

(Note that this last argument is a fairly standard synthesis of “two ends” type conditions with the Bourgain bush argument.

From this point on we will essentially forget that E_2'' came from the multilinear distance problem. We will use only the nondegeneracy of E_2'' together with the conclusion of Lemma 1.8.)

§2 ESTABLISHING ADDITIVE AND MULTIPLICATIVE STRUCTURE

In this section, we begin with a set E_2'' which is nondegenerate and so that

$$(2.1) \quad |\{(x_1, x_2) \in (E_2'')^2 : x_1 \triangle x_2\}| \gtrsim \delta^d.$$

We begin by discretizing the set of lines implicit in the inequality (2.1). In what follows, we define a δ -tube to be the set of points at distance $\lesssim \delta$ from a line. (This is a little thicker than what is traditionally meant by a δ -tube.)

Lemma 2.1 There is a set \mathcal{T} of δ -separated δ -tubes with the property that

$$\#(\mathcal{T}) \gtrsim \delta^{1-d},$$

while

$$\sum_{T \in \mathcal{T}} |T \cap E_2''| \gtrsim \delta^{\frac{1}{2}}.$$

Proof We begin by stating an immediate consequence of nondegeneracy (Lemma 1.2) which will be used frequently in this section.

Lemma 2.2 Let T be a δ -tube, then

$$|T \cap E''| \lesssim \delta^{d-\frac{1}{2}}.$$

Next we observe that for any $x_1, x_2 \in E''$ with $x_1 \triangle x_2$, there is a δ -tube $T_{x_1 x_2}$ containing x_1 and x_2 so that

$$|T_{x_1 x_2} \cap E''| \gtrsim \delta^{d-\frac{1}{2}}.$$

We pick a maximal δ -separated set of pairs (x_1, x_2) and define \mathcal{T}_0 to be the associated tubes. By (2.1), we have that

$$\#(\mathcal{T}_0) \gtrsim \delta^{-d}.$$

Next we pick a maximal δ -separated set of tubes in \mathcal{T}_0 and refer to it as (\mathcal{T}) . By Lemma 2.2, we have that

$$\#(\mathcal{T}) \gtrsim \delta^{1-d}.$$

□

We observe that for any $x \in E''$, we have that

$$(2.2) \quad \#(\{T \in \mathcal{T} : x \in T\}) \lesssim \delta^{\frac{d-1}{2}},$$

since otherwise, we would have, by Bourgain's bush argument, that $|E''| \gg \delta^{\frac{d}{2}}$, which is impossible by nondegeneracy.

We define F to be the set of $x \in E''$ so that

$$\#(\{T \in \mathcal{T} : x \in T\}) \gtrsim \delta^{\frac{d-1}{2}}.$$

We record an immediate consequence of (2.2), Lemma 2.1 and Chebychev's inequality :

Lemma 2.3

$$|F| \gtrsim |E''| \gtrsim \delta^{\frac{d}{2}}.$$

We next observe that any refinement of F will still satisfy the conclusion of Lemma 2.1

Lemma 2.4 Let $G \subset F$ with $|G| \gtrsim |F|$ then

$$\sum_{T \in \mathcal{T}} |T \cap G| \gtrsim \delta^{\frac{1}{2}}.$$

Proof Simply integrate $\#(\{T \in \mathcal{T} : x \in T\})$ over x in G . □

We now find a refinement of F which after a suitable projective transformation is contained in a product set.

For $x, y \in F$, we define $x \square y$ provided $|x - y| \gtrsim 1$ and there is $T \in \mathcal{T}$ with $x, y \in T$. We recall that y_1, \dots, y_d are in essentially general position (*EGP*) if

$$|(y_2 - y_1) \wedge \dots \wedge (y_d - y_1)| \gtrsim 1.$$

Lemma 2.5 For $y_1, \dots, y_d \in F$, let

$$F(y_1, \dots, y_d) = \{x \in F : x \square y_1, y_2, \dots, y_d\}.$$

Then

$$|\{(x, y_1, \dots, y_d) : x \in F(y_1, \dots, y_d), y_1, \dots, y_d \in EGP\}| \gtrsim \delta^{\frac{d(d+1)}{2}}.$$

Proof We let \tilde{F} be the set of x in F with $|\{y \in F : x \square y\}| \gtrsim \delta^{\frac{d}{2}}$. In light of nondegeneracy and Lemma 2.4, we have that $|\tilde{F}| \gtrsim |F|$. Thus we conclude

$$|\{x, y \in F : x \square y\}| \gtrsim \delta^d.$$

Applying Hölder's inequality (Lemma 1.4 (iii)), we get that

$$(2.3) \quad |\{x, y_1, \dots, y_d \in F : x \square y_1, \dots, y_d\}| \gtrsim \delta^{\frac{d(d+1)}{2}}.$$

Now applying nondegeneracy we get

$$(2.4) \quad |\{x, y_1, \dots, y_d \in F : x \square y_1, \dots, y_d, |(y_2 - y_1) \wedge \dots \wedge (y_d - y_1)| \ll 1\}| \ll \delta^{\frac{d(d+1)}{2}}.$$

Comparing (2.3) with (2.4) gives the desired result. \square

Applying Chebychev's inequality, we may find $y_1, \dots, y_d \in F$ in essentially general position so that $|F(y_1, \dots, y_d)| \gtrsim |F|$. We let F' be the set of $x \in F(y_1, \dots, y_d)$ which are at distance $\gtrsim 1$ from the hyperplane containing y_1, \dots, y_d . By nondegeneracy, we have that $|F'| \gtrsim |F|$.

Next we find a suitable projective transformation. We imbed \mathbb{R}^d in \mathbb{RP}^d by sending (z_1, \dots, z_d) to $(z_1, \dots, z_d, 1)$. We define f_j , the j th cardinal point at ∞ to be $(e_j, 0)$ where e_j is the j th standard basis vector in \mathbb{R}^d . We let P be a (linear) projective transformation taking y_j to f_j . Observe that P is Lipschitz when restricted to F' with biLipschitz constant $\lesssim 1$.

We introduce some notation. Let

$$\pi_j : \mathbb{R}^d \longrightarrow \mathbb{R},$$

be the rank 1 projection which maps a point in \mathbb{R}^d to its j th coordinate. Let

$$\pi^j : \mathbb{R}^d \longrightarrow \mathbb{R}^{d-1},$$

be the rank $d-1$ projection which maps a point in \mathbb{R}^d to the point omits the j th coordinate.

By construction, we know that for each j , we have

$$|\pi^j(P(F'))| \lesssim \delta^{\frac{d-1}{2}},$$

for each j . We would like to mimic the argument in [KT]. In order to do this we would need

$$|\pi_j(P(F'))| \lesssim \delta^{\frac{1}{2}}.$$

Fortunately, we may achieve this by picking a suitable refinement of F' . (This is the only step which has no analog in the two dimensional case because $1 + 1 = 2$.)

Lemma 2.6 Let $A \subset \mathbb{R}^d$ with

$$|A| \gtrsim \delta^{\frac{d}{2}}.$$

Suppose that for all j , we have that

$$|\pi^j(A)| \lesssim \delta^{\frac{d-1}{2}}.$$

(Note that this immediately implies $|A| \lesssim \delta^{\frac{d}{2}}$.) Then we may find $A' \subset A$ with

$$|A'| \gtrsim |A|,$$

so that for each j , we have

$$|\pi_j(A)| \lesssim \delta^{\frac{1}{2}}.$$

Proof We need only find a refinement A' with $|\pi_1 A| \lesssim \delta^{\frac{1}{2}}$ because we can then permute the coordinates and repeat the procedure. Our main idea will be to observe that the map π_1 factors through the maps π^2, \dots, π^d . We define $A_2 \subset A$ with $|A_2| \gtrsim |A|$ and so that for each $y \in \pi^2(A_2)$, we have $|\pi_2((\pi^2)^{-1}(y) \cap A)| \gtrsim \delta^{\frac{1}{2}}$. Then we define $A_3 \subset A$ with $|A_3| \gtrsim |A_2|$ and so that for each $y \in \pi^3(A_2)$, we have $|\pi_3(\pi^3)^{-1}(y) \cap A_2| \gtrsim \delta^{\frac{1}{2}}$. We continue in the same fashion until we have defined $A_d = A'$ so that for each $y \in \pi^d(A_d)$,

we have $|\pi_d(\pi^d)^{-1}(y) \cap A_{d-1}| \gtrsim \delta^{\frac{1}{2}}$. Then since π_1 factors through each of π^j for $j \geq 2$, by construction, we have for each $x \in A_d$ that

$$|\pi^1(\pi_1^{-1}(\pi_1(x)) \cap A| \gtrsim \delta^{\frac{d-1}{2}},$$

so that by the upper bound on $|A|$, we have

$$|\pi_1(A_d)| \lesssim \delta^{\frac{1}{2}}.$$

□

Applying Lemma 2.6 to $P(F')$, mapping back under P^{-1} and invoking the biLipschitz property of P , we see that we may choose $F'' \subset F'$ with

$$|F''| \gtrsim |F'|,$$

so that

$$P(F'') \subset A_1 \times A_2 \times \cdots \times A_d,$$

with each A_j a $(\delta, \frac{1}{2})_1$ set. (To obtain the $(\delta, \frac{1}{2})_1$ property, observe that by the large amount of incidence with elements of \mathcal{T} , there is a line whose intersection with F' is biLipschitz to refinements of each of A_1, \dots, A_d . Then we can invoke the nondegeneracy of F' . We leave the details to the reader.)

We are almost done. Our goal is now to show that A_d has good additive and multiplicative properties, thereby contradicting Bourgain's sum product estimate. Our arguments now follow [KT] essentially verbatim.

We record a version (*cf* [KT]) of the Gowers-Balog-Szemerédi theorem which we shall use.

Lemma 2.7 Let $A, B \subset \mathbb{R}$ be bounded $(\lesssim 1)$ $(\delta, \frac{1}{2})_1$ sets with

$$|A|, |B| \gtrsim \delta^{\frac{1}{2}}.$$

Suppose $G \subset A \times B$ with

$$|G| \gtrsim \delta,$$

so that

$$|\{a + b : (a, b) \in G\}| \lesssim \delta^{\frac{1}{2}},$$

then there exists $A' \subset A$ with

$$|A'| \gtrsim |A|,$$

and

$$|A' - A'| \lesssim |A|.$$

We also record a version of Bourgain's theorem [B] which will provide us with our contradiction:

Theorem 2.8(Bourgain) Let $A \subset \mathbb{R} \setminus \{0\}$ with A and $\frac{1}{A}$ bounded and with A a $(\delta, \frac{1}{2})_1$ set satisfying

$$|A| \lesssim \delta^{\frac{1}{2}}.$$

Then either

$$|A - A| \gg \delta^{\frac{1}{2}},$$

or

$$|AA| \gg \delta^{\frac{1}{2}}.$$

We now recall where we are.

We have $(\delta, \frac{1}{2})_1$ sets A_1, \dots, A_d so that $P(F'') \cap A_1 \times \dots \times A_d$ is a refinement of $A_1 \times \dots \times A_d$. We may pick a refinement $A' \subset A_d$ with $|A'| \gtrsim |A_d|$ so that for any $A'' \subset A'$ with $|A''| \gtrsim |A_d|$ we still have

$$|P(F'') \cap A_1 \times \dots \times A''| \gtrsim |A_1 \times \dots \times A_d|.$$

This will allow us to refine A_d twice, once to ensure additive properties and once to ensure multiplicative properties.

Most importantly, a large number of lines have large intersection with $A' \times \dots \times A_d$. This is because of the projective and Lipschitz properties of the map P . This if we apply P to the part of T at distance ≈ 1 from the plane spanned by y_1, \dots, y_d , where $T \in \mathcal{T}$, we get a subset of a δ -tube. This gives us a lot of arithmetic identities.

To be precise, let Z consist of the set of all pairs $(a_{11}, \dots, a_{1d}), (a_{21}, \dots, a_{2d})$ in $A_1 \times \dots \times A_d$ and all elements $a_{01} \in A_1$ with the property that there exist a_{02}, \dots, a_{0d} in A_2, \dots, A_d respectively with

$$(2.5) \quad a_{1j} \left[\frac{a_{01} - a_{11}}{a_{21} - a_{11}} \right] + a_{2j} \left[\frac{a_{21} - a_{01}}{a_{21} - a_{11}} \right] \sim a_{0j},$$

for all j from 2 to d . Then

$$|Z| \gtrsim \delta^{\frac{2d+1}{2}}.$$

We fix $j = d$. Then appropriately fixing all variables but a_{1d} and a_{2d} in (2.5) we see that we can find $A' \subset A_d$ so that

$$(2.6) \quad |A_d - A_d| \lesssim \delta^{\frac{1}{2}},$$

using Lemma 2.7. Indeed we can do that same to A_1 . We observe that we obtain many equations of the form (2.5) running over only variables from the refined sets.

Then we fix all variables but a_{1d} and a_{01} . We obtain many equations of the form

$$C_1 a_{1d} a_{01} + C_2 a_{01} \sim a_{0d}.$$

Thus we may find refinements $A'' \subset A'$ so that

$$|(A'' + \frac{C_2}{C_1})(A'' + \frac{C_2}{C_1})| \lesssim \delta^{\frac{1}{2}},$$

applying Lemma 2.7 multiplicatively (that is, to the logarithms) However by (2.6), we see that

$$|(A'' + \frac{C_2}{C_1}) + (A'' + \frac{C_2}{C_1})| \lesssim \delta^{\frac{1}{2}},$$

so that we have contradicted Theorem 2.8. Thus our assumption is false, and we have proven Theorem 1.1. \square

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